Decoding in the Rank Metric: from Finite Fields to Finite Rings

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(This Talk is based on a work done with H. T. Kamche)

Plan de la présentation

Next

Linear code

1. $(\mathrm{R}^n, \|\cdot\|)$, R a finite field/ring and $\|\cdot\|$ a norm

2. Linear code $\mathscr{C}$ = free.sm of $(\mathrm{R}^n, \|\cdot\|)$

$$\mathscr{C} = \bigoplus_{i=1}^{k} \mathrm{R}\ \vec{v}_i$$

where $\vec{v}_i$ are linearly independent.

3. The matrix $\boldsymbol{G} = \begin{pmatrix} \vec{v}_1 \\ \cdot \\ \cdot \\ \cdot \\ \vec{v}_k \end{pmatrix}$ is called a generator matrix of $\mathscr{C}$

4. Decoding a word $\vec{w} \in \mathrm{R}^n$ = solving the closest vector problem (CVP)

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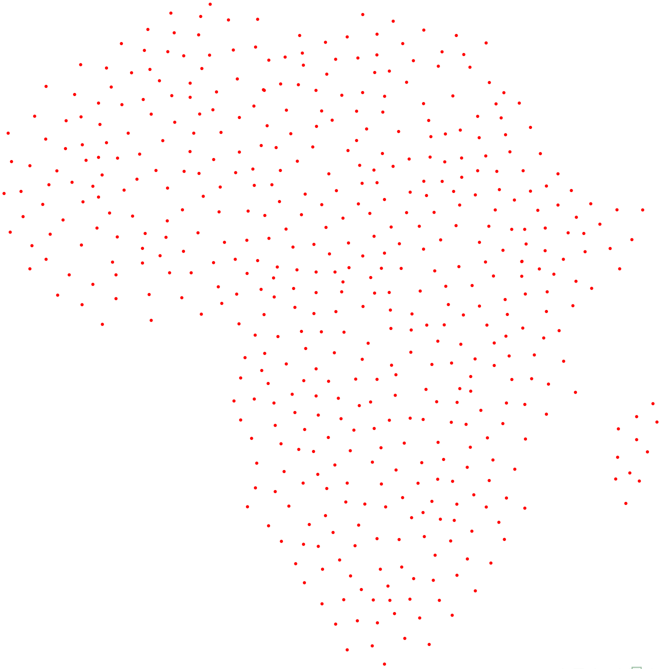
General Decoding Problem = Closest Vector Problem (CVP)



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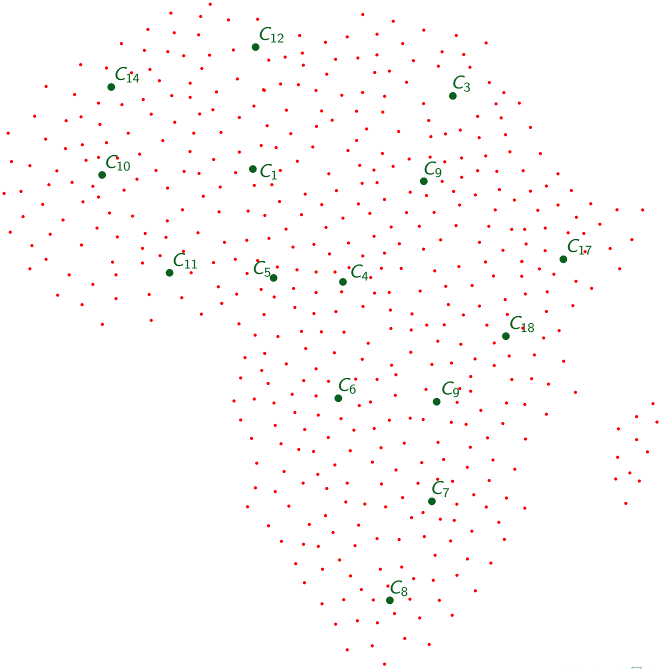

Hardness of decoding

- Decoding is NP-Hard for a "random" linear code
- For Hamming metric: Berlekamp-McEliece-Van Tilborg '78

Solving the decoding problem

- Information set decoding
- Introduced by Prange '62
- Complexity: $2^{at(1+o(1))}$

$$a = constante\left(\frac{k}{n}, \frac{t}{n}\right)$$

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McEliece Cryptosystem ('78)

1. Based on linear codes equipped with an efficient decoding algorithm
   - Public key = random basis
   - Private key = decoding algorithm (good basis)
2. McEliece proposed binary Goppa codes

Security assumptions

- Indistinguishability of Goppa codes **Courtois-Finiasz-Sendrier '01**
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2. No efficient attack
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Rank metric and Hamming metric

Let $\mathbb{F}_{q^m}/\mathbb{F}_q$ and $\vec{x} = (x_1 \cdots x_n) \in \mathbb{F}_{q^m}^n \equiv \mathbb{F}_q^{m \times n}$.

$$\|\vec{x}\|_h = \#\{\ i\ :\ x_i \neq 0\}$$

$$\|\vec{x}\|_q = \dim\langle x_1, \cdots, x_n\rangle_{\mathbb{F}_q}$$

Example

- $\mathbb{F} = \mathbb{F}_{2^5} = \mathbb{F}_2 < w > = < 1, w, w^2, w^3, w^4 >_{\mathbb{F}_2}$
- $\vec{x} = (w, 0, 0, w)$

1. **Hamming metric:**
   - $\|\vec{x}\|_h = 2$
2. **Rank metric:**
   - $\|\vec{x}\|_2 = \dim\left(< w, w >_{\mathbb{F}_2}\right) = 1$

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Metric	Approach	Complexity
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Rank	• Combinatorial attacks • Ourivski–Johannsson (2002)	$2^{tn+o(1)}$

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Rank-Based Cryptography

First Rank-Metric Encryption Scheme

- First proposal from at Eurocrypt'91 : GPT Cryptosystem
- Broken by Overbeck at Mycrypt'05

Recent Proposals

- New proposal at WCC'13 based on LRPC codes[a] (LDPC in the rank metric)
- Submissions to the NIST PQ competition (ROLLO, RQC)
  - ROLLO : Analogue of NTRU, uses LRPC codes
  - RQC : Security relying only on the CVP in the rank metric
  - Shorter public keys

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Rank Metric : Improvement of Algebraic Attacks in 2020

Solving the decoding problem in the rank metric

1. - Combinatorial attacks
   - Aragon-Gaborit-Hautville-Tillich '18

   $$2^{tn+o(1)}$$

2. - New algebraic attack
   - By Bardet et al. Eurocrypt'20

   $$2^{O(t\log_2(n))}$$

Consequenses

- Drastic reduction of security levels
- ROLLO−I−128/192/256 $\rightsquigarrow$ ROLLO−I−71/87/151
- RQC−256 $\rightsquigarrow$ RQC−188
- Elimination of ROLLO and RQC from the NIST competition

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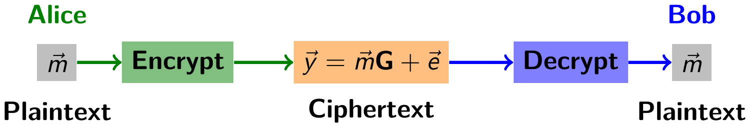
Excerpt from the NIST Report on the Second Round of the PQCS

"... Despite the development of algebraic attacks, NIST believes rank-based cryptography should continue to be researched. The rank metric cryptosystems offer a nice alternative to traditional hamming metric codes with comparable bandwidth... " [1]

[1] Status Report on the Second Round of the NIST Post-Quantum Cryptography Standardization Process, **July 2020**

Starting Point for all Algebraic Attacks

- $\mathscr{C}$ is a $(n,k)_{\mathbb{F}_{q^m}}-$code generated by $\boldsymbol{G}$



Ourivski-Johannsson's Modelling

- $\mathscr{C}_{ext}$ the $(n,k+1)-$code generated by

$$\mathscr{C}_{ext} = < \begin{pmatrix} \boldsymbol{G} \\ \vec{y} \end{pmatrix} >_{\mathbb{F}_{q^m}} = < \begin{pmatrix} \boldsymbol{G} \\ \vec{m}\boldsymbol{G} + \vec{e} \end{pmatrix} >_{\mathbb{F}_{q^m}} = < \begin{pmatrix} \boldsymbol{G} \\ \vec{e} \end{pmatrix} >_{\mathbb{F}_{q^m}}$$

$$\Longrightarrow \exists \vec{c}\,' \in \mathscr{C}_{ext} \text{ s.t } \mathrm{Rank}_{\mathbb{F}_q}(\vec{c}\,') = r$$

- Each elt of the form $\lambda\vec{e}$, $\lambda \in \mathbb{F}_{q^m}^*$ is a good candidate

Starting Point of Recent Algebraic Attacks

- $\mathscr{C}$ is a $(n,k)_{\mathbb{F}_{q^m}}$ −code generated by $\boldsymbol{G}$

Alice — **Bob**

$\vec{m}$ → **Encrypt** → $\vec{y} = \vec{m}\mathbf{G} + \vec{e}$ → **Decrypt** → $\vec{m}$

Plaintext — **Ciphertext** — **Plaintext**

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- $\mathscr{C}$ is a $(n, k)_S-$code generated by $\boldsymbol{G}$

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Rank Decoding Problem over Finite Rings

Illustration with words viewed as matrices

- Let $R = \mathbb{Z}_6$ and $\boldsymbol{A} = \begin{pmatrix} 5 & 0 \\ 0 & 3 \end{pmatrix}$. $2\boldsymbol{A} = \begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix}$.
- We have

$$\mathrm{Rank}_{R}(\boldsymbol{A}) = 2, \text{ while } \mathrm{Rank}_{R}(2\boldsymbol{A}) = 1$$

- Ourivski-Johannsson's modelling is not applicable[2]
- All known algebraic attacks/costs are not applicable

[2] Kalachi & Kamche. On the rank decoding problem over finite principal ideal rings. *AMC'23*

Next

Finite Chain Rings

- A finite ring R is a finite chain ring if the set of its ideals forms a chain.
  * e.g. $\mathbb{Z}_{p^k} = \mathbb{Z}/p^k\mathbb{Z}$ with $k \in \mathbb{N}^*$ and p a prime number.

$$\{0\} \subset p^{k-1}\mathbb{Z}_{p^k} \subset p^{k-2}\mathbb{Z}_{p^k} \subset \cdots \subset p\mathbb{Z}_{p^k} \subset \mathbb{Z}_{p^k}$$

- Given a finite chain ring R, // e.g. $\mathbb{Z}_8 = \mathbb{Z}_{2^3} = \mathbb{Z}/2^3\mathbb{Z}$,
- $\mathfrak{m} = \pi R$ is the maximal ideal of R // $\mathfrak{m} = 2\mathbb{Z}_{2^3}$
- $\mathbb{F}_q = R/\mathfrak{m}$ is the residue field of R // $\mathbb{Z}_{2^3}/2\mathbb{Z}_{2^3} = \mathbb{F}_2$
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Galois Extension of a FCR : $\mathbb{Z}_8$

Let us take $R = \mathbb{Z}_8 = \mathbb{Z}_{2^3}$ and consider the polynomial

$$h(X) = X^3 + 6X^2 + 3X + 1 \in R[X].$$

Its projection modulo 2 is

$$\Psi(h)(X) = X^3 + X + 1,$$

which is irreducible in $\mathbb{F}_2[X]$.

Consider the ring

$$S = \mathbb{Z}_8[X]/(h(X))$$

S is said to be a Galois extension of degree 3 of $\mathbb{Z}_8$.

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Rank Metric

Definition 1

Let S be a Galois extension of degree m over a finite chain ring R. Let $\beta = (\beta_1, \ldots, \beta_m)$ be an R-basis of S. Any vector $\vec{x} = (x_1, \ldots, x_n) \in S^n$ can be uniquely represented as a matrix over R:

$$\vec{x} \longleftrightarrow \boldsymbol{X} = (X_{ij}) = \begin{pmatrix} X_{11} & X_{12} & \cdots & X_{1n} \\ X_{21} & X_{22} & \cdots & X_{2n} \\ \vdots & \vdots & & \vdots \\ X_{m1} & X_{m2} & \cdots & X_{mn} \end{pmatrix} \in R^{m \times n},$$

So that $\vec{x} = \beta \boldsymbol{X}$. The **rank of** $\vec{x}$, denoted $\mathrm{Rank}_R(\vec{x})$, is defined as:

$$\mathrm{Rank}_R(\vec{x}) := \mathrm{rank}_R(\boldsymbol{X}).$$

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Rank Metric

Example

Consider the finite chain ring $R = \mathbb{Z}_4$ and the extension

$$S = \frac{\mathbb{Z}_4[X]}{(X^3 + 3X + 1)} = \frac{\mathbb{Z}_4[X]}{(f(X))},$$

of degree $m = 3$ over R.

- $a := X \mod f(X)$
- $S = R[a] = \langle 1, a, a^2 \rangle_R$.
- $\vec{x} = (1 + 2a + 3a^2,\ 2 + 2a^2,\ 2a,\ 0) \in S^4$

Its matrix representation over R is:

$$\boldsymbol{X} = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 2 & 0 & 2 & 0 \\ 3 & 2 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Thus, $\mathrm{Rank}_R(\vec{x}) = 2$.

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Next

Rank Decoding Problem over Finite Chain Rings

- R is a finite chain ring
- S is a Galois extension of R

Definition 2 (Rank Decoding Problem $\mathcal{RD}$)

- Let $\mathcal{C}$ be a (n, k)-linear code over S
- $\vec{y} \in S^n$ and $t \in \mathbb{N}^*$

The *Rank Decoding Problem* is to find $\vec{e} \in S^n$ and $\vec{c} \in \mathcal{C}$ such that :

$$\vec{y} = \vec{c} + \vec{e} \text{ and } \mathtt{Rank}_R(\vec{e}) \leqslant t$$

- Let $\mathbf{H} \in S^{(n-k)\times n}$, be a parity check matrix of $\mathcal{C}$

$$\text{i.e. } \forall \vec{c} \in \mathcal{C},\ \vec{c}\boldsymbol{H}^\top = \vec{0}$$

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$$\vec{y}\boldsymbol{H}^\top = (\vec{c} + \vec{e})\boldsymbol{H}^\top = \vec{c}\boldsymbol{H}^\top + \vec{e}\boldsymbol{H}^\top = \vec{e}\boldsymbol{H}^\top$$

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Rank Syndrome Decoding Problem over Finite Chain Rings

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Definition 3 (Rank Syndrome Decoding Problem $\mathcal{RSD}$)

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Rank Syndrome Decoding Problem (example)

Setup

- Let $R = \mathbb{Z}_8$ and $S = R[a]$, where a is a root of the polynomial $h(x) = x^3 + 6x^2 + 3x + 1$.
- Then $\mathbf{b} = (1, a, a^2)$ forms an R-basis of S.
- Consider the $[3, 1, 3]$-linear code $\mathcal{C} \subset S^3$ generated by the matrix:

$$\boldsymbol{G} = \begin{pmatrix} 1 & a^2 + 5 & a^2 + a + 1 \end{pmatrix}$$

- Let

$$\vec{y} = (6a^2 + 3,\ \ 5a^2 + 1,\ \ 3a^2 + a + 7) \in S^3$$

- A parity check matrix of $\mathscr{C}$ is

$$\boldsymbol{H} = \begin{pmatrix} 7a^2 + 3 & 1 & 0 \\ 7a^2 + 7a + 7 & 0 & 1 \end{pmatrix}$$

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- *Let S be a Galois extension of degree m of R*
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There exists a free submodule $F \subset S$ such that

$$supp(\vec{e}) = \langle e_1, \cdots, e_n \rangle_R \subset F \text{ and } \mathtt{Rank}_R(F) = u$$

Example (Keeping the previous setting)

Let $\vec{e} = (6a^2 + 2, \quad 4a^2 + 4, \quad 2a^2 + 6) \in S^3$, with $S = R[a]$ and $R = \mathbb{Z}_8$.

- We have $\mathtt{Rank}_R(\vec{e}) = 1$ and $m = 3$
- For $u = 2$, set

$$F = \langle 1, \quad a^2 \rangle_R$$

- One can see that F is free with rank 2 and $supp(\vec{e}) \subset F$.

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Problem Reminder

Setup

- Let $R = \mathbb{Z}_8$ and $S = R[a]$, where a is a root of the polynomial $h(x) = x^3 + 6x^2 + 3x + 1$.
- Then $\mathbf{b} = (1, a, a^2)$ forms an R-basis of S.
- Consider the $[3, 1, 3]$-linear code $\mathcal{C} \subset S^3$ generated by the matrix:

$$\boldsymbol{G} = \begin{pmatrix} 1 & a^2 + 5 & a^2 + a + 1 \end{pmatrix}$$

- Let

$$\vec{y} = \begin{pmatrix} 6a^2 + 3, & 5a^2 + 1, & 3a^2 + a + 7 \end{pmatrix} \in S^3$$

- A parity check matrix of $\mathscr{C}$ is

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- Let $u \in \mathbb{N}$ such that $t \leqslant u \leqslant m$ $\quad //t = 1$ and $m = 3$. So we choose $u = 2$
- F a rank u free $R-$s.m of S s.t $supp(\vec{e}) \subset F$ $\quad //$Let say $F = \langle 1, a^2 \rangle_R$
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- The system $\mathscr{E}_1$ to be solved is then,

$$\left(\begin{array}{cc|cc|cc} 7a^2 + 3 & 2a^2 + 7a + 2 & 1 & a^2 & 0 & 0 \\ 7a^2 + 7a + 7 & 4a^2 + 2a + 3 & 0 & 0 & 1 & a^2 \end{array}\right) \begin{pmatrix} x_{11} \\ x_{21} \\ \hline x_{12} \\ x_{22} \\ \hline x_{13} \\ x_{23} \end{pmatrix} = \begin{pmatrix} 6a^2 + 2a + 6 \\ 2a + 6 \end{pmatrix}$$

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- By expanding $\mathscr{E}_1$ in R, we get the linear system $\mathscr{E}_2$ given by

$$\left(\begin{array}{cccccc} 3 & 2 & 1 & 0 & 0 & 0 \\ 0 & 7 & 0 & 0 & 0 & 0 \\ 7 & 2 & 0 & 1 & 0 & 0 \\ \hline 7 & 3 & 0 & 0 & 1 & 0 \\ 7 & 2 & 0 & 0 & 0 & 0 \\ 7 & 4 & 0 & 0 & 0 & 1 \end{array}\right)\begin{pmatrix} x_{11} \\ x_{21} \\ \hline x_{12} \\ x_{22} \\ \hline x_{13} \\ x_{23} \end{pmatrix} = \left(\begin{array}{c} 6 \\ 2 \\ 6 \\ \hline 6 \\ 2 \\ 0 \end{array}\right)$$

- $\mathscr{E}_2$ has $(n-k)\times m$ equations and $u \times n$ unknowns.
- So, a nescessary condition to have at most one solution is $(n-k)\times m \geqslant u \times n$. That is to say, $u \leqslant \lfloor (n-k)m/n \rfloor$
- The max value for u is 2 in our example.

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Solving the RSD Problem over FCR.

- The system $\mathscr{E}_1$ is

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Principal Ideal Rings $\longrightarrow$ Finite Chain Rings

Theorem 5 (McDonald, 1974)

If R is a principal ring, then

$$A \cong A_{(1)} \times \cdots \times A_{(\rho)}$$

where each $A_{(j)}$ is a finite chain ring.

Ring of Integers Modulo n (CRT)

- Let $p_1, \ldots, p_d$ be distinct prime numbers and $k_1, \ldots, k_d \in \mathbb{N}^*$,
- Define $n = p_1^{k_1} \cdots p_d^{k_d}$

Then:

$$\mathbb{Z}_n \cong \mathbb{Z}_{p_1^{k_1}} \times \cdots \times \mathbb{Z}_{p_d^{k_d}}$$

Conclusion

Key Takeaways

- The **rank metric** is a powerful alternative to the classical Hamming metric.
- Recent algebraic attacks motivate to explore **finite rings**, where *torsion* breaks field-based assumptions.
- We analyzed **combinatorial attacks**[a] on the RSD problem over finite principal ideal rings.
- Complexity increases due to ring structure: the **nilpotency index** ν appears in the exponent.

[a] Gaborit, Ruatta & Schrek. On the complexity of the rank syndrome decoding problem. *IEEE IT'16*

Modeling Strategy	Over Fields	Over Rings
First Modeling	$\mathcal{O}\left(m^3(n-k)^3 q^{rk}\right)$	$\mathcal{O}\left(m^3(n-k)^3 q^{\nu rk}\right)$
Second Modeling	$\mathcal{O}\left(m(n-k)u^2 n^2 q^{r\lfloor \frac{mk}{n} \rfloor}\right)$	$\mathcal{O}\left(m(n-k)u^2 n^2 q^{\nu r\lfloor \frac{mk}{n} \rfloor}\right)$

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Perspectives

Research Directions

- Explore **improvements** of existing combinatorial algorithms over finite rings.

- Several enhancements exist for combinatorial attacks,
  - but they all rely on the **Ourivski-Johannsson** model,
  - which **fails in the presence of torsion**.

- Developping an **efficient algebraic modeling over rings** remains an open problem

- *First steps* in this direction were initiated by **Kamche and Kalachi**[a].

- These studies could have an impact on several other problems such as MinRank and VSF[b]

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  - ➠ which **fails in the presence of torsion**.
- Developping an **efficient algebraic modeling over rings** remains an open problem
- *First steps* in this direction were initiated by **Kamche and Kalachi**[a].
- These studies could have an impact on several other problems such as MinRank and VSF[b]

[a] Kamche & Kalachi. Solving systems of algebraic equations over finite commutative rings and applications. *AAECC'24*

[b] P. Gaborit, M. Haiech & R. Neveu. A digital signature scheme based on the vector space factorization problem and the MPC-in-the-Head paradigm. *AMC'25*

$\Longrightarrow$ *Extending rank decoding theory beyond fields is a rich and promising area.*